

Systemic losses in banking networks: indirect interaction of nodes via asset prices

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Abstract

A simple banking network model is proposed which features multiple waves of bank defaults and is analytically solvable in the limiting case of an infinitely large homogeneous network. The model is a collection of nodes representing individual banks; associated with each node is a balance sheet consisting of assets and liabilities. Initial node failures are triggered by external correlated shocks applied to the asset sides of the balance sheets. These defaults lead to further reductions in asset values of all nodes which in turn produce additional failures, and so on. This mechanism induces indirect interactions between the nodes and leads to a cascade of defaults. There are no interbank links, and therefore no direct interactions, between the nodes. The resulting probability distribution for the total (direct plus systemic) network loss can be viewed as a modification of the well-known Vasicek distribution.

The purpose of this short note is to introduce a banking network model capable of describing cascading bank failures. Consider a banking system where each bank is represented as a node. Each node i has an associated balance sheet which consists of assets A_i and liabilities L_i . Nodes are solvent if $A_i > L_i$, and they default when this inequality no longer holds as a result of the stressed asset side. All nodes are assumed to be initially solvent. The first wave of defaults is triggered by external correlated shocks applied to the asset sides of the balance sheets. The shocks change the asset values from A_i to new values $A_{i,1}$. Some of them go below the liabilities ($A_{i,1} < L_i$), and the corresponding nodes fail. These defaults lead to extra shocks for assets, changing the asset values from $A_{i,1}$ to $A_{i,2}$. As a result, some of the nodes which survived the initial shocks fail to satisfy the new solvency condition $A_{i,2} > L_i$, and default. The second default wave produces yet another set of shocks for asset prices, more nodes fail, and so on. All but the initial shocks are modelled by discounting asset prices $A_{i,1}$, and the discount factor applied to $A_{i,1}$ after k default waves has the form

$$D_k = \exp(-aq_k) \quad (1)$$

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where q_k is the network loss at this stage defined as the fraction of defaulted nodes, and a is a constant [1-3]. Asset prices $A_{i,1}$ are assumed to be lognormal; the asset log changes (returns)

$$R_i = \ln \frac{A_{i,1}}{A_i} \quad (2)$$

are jointly normally distributed with identical means μ and covariances $\sigma^2 \rho$, where σ is the standard deviation and ρ is the correlation coefficient. These variables admit the representation

$$R_i = \mu + \sigma x_i, \quad x_i = \sqrt{\rho} Z + \sqrt{1-\rho} \varepsilon_i \quad (3)$$

where Z is the market factor common to all nodes, and ε_i are the node-specific idiosyncratic factors. Variables x_i are jointly standard normal with correlations ρ , while Z and all ε_i are standard normal and mutually independent (e.g., [4]). It is convenient for what follows to rewrite Eq. (3) as

$$R_i = \alpha \varepsilon_i + \beta \quad (4)$$

where

$$\alpha = \sigma \sqrt{1-\rho}, \quad \beta = \mu + \sigma \sqrt{\rho} Z \quad (5)$$

The rest of the exposition is focused on the limiting case of an infinitely large ($n \rightarrow \infty$, where n is the number of nodes) and homogeneous ($A_i = A$ and $L_i = L$ for any i) network in which the full analytical solution is possible.

The equation describing the cascade of node defaults in this limiting case can be derived as follows. The node i fails in the initial default wave if $A_{i,1} < L$. This corresponds, via Eqs. (2)-(5), to

$$\varepsilon_i < \delta_1, \quad \delta_1 = \frac{1}{\alpha} \left(\ln \frac{L}{A} - \beta \right) \quad (6)$$

The network loss q_1 at this stage, which is the direct loss caused by external shocks and contains no systemic contribution, is equal to the probability of the outcome $A_{i,1} < L$. Since ε_i is a standard normal variable, the fraction q_1 is given by

$$q_1 = P(A_{i,1} < L) = P(\varepsilon_i < \delta_1) = N(\delta_1) \quad (7)$$

where $P(E)$ denotes the probability of the event E , and $N(x)$ is the standard normal cumulative density function (CDF). The idiosyncratic loss q is equal to the probability of an individual default and corresponds to vanishing correlation $\rho = 0$. The relation between the direct loss q_1 and the idiosyncratic loss q is given by

$$\delta_1 = \frac{N^{-1}(q) - \sqrt{\rho} Z}{\sqrt{1-\rho}} \quad (8)$$

together with Eq. (7). The first wave of defaults changes asset values from $A_{i,1}$ to

$$A_{i,2} = D_1 A_{i,1} = A_{i,1} \exp(-a q_1) \quad (9)$$

triggering the second default wave. The network loss increases from q_1 to

$$q_2 = P(A_{i,2} < L) = P(\varepsilon_i < \delta_2) = N(\delta_2) \quad (10)$$

where

$$\delta_2 = \frac{1}{\alpha} \left(\ln \frac{L}{A} - \beta + a q_1 \right) = \delta_1 + \kappa N(\delta_1) \quad (11)$$

and

$$\kappa = \frac{a}{\alpha} = \frac{a}{\sigma \sqrt{1-\rho}} \quad (12)$$

After the second default wave the asset values become

$$A_{i,3} = D_2 A_{i,1} = A_{i,1} \exp(-a q_2) \quad (13)$$

and the network loss increases to

$$q_3 = P(A_{i,3} < L) = P(\varepsilon_i < \delta_3) = N(\delta_3) \quad (14)$$

where now

$$\delta_3 = \frac{1}{\alpha} \left(\ln \frac{L}{A} - \beta + a q_2 \right) = \delta_1 + \kappa N(\delta_2) \quad (15)$$

This process continues to infinity. After k default waves the network loss is

$$q_k = P(A_{i,k} < L) = P(\varepsilon_i < \delta_k) = N(\delta_k) \quad (16)$$

and the cascade equation has the form

$$\delta_k = F(\delta_{k-1}), \quad F(x) = \delta_1 + \kappa N(x) \quad (17)$$

Eq. (17) is a one-dimensional iterated map, the set $\{x, F(x), F(F(x)), \dots\}$ is called the orbit of x under F , and x is the initial value of the orbit (e.g., [5, 6]). Function $F(x)$ is continuous and increasing, so Eq. (17) is an invertible map and can only have fixed points. When more than one fixed point exists, they are alternatively stable and unstable, and the unstable fixed points are the boundaries that separate the basins of attraction of the stable fixed points (e.g., [7]).

The total loss q_∞ is the fraction of failed nodes after the cascade of defaults exhausts itself, i.e., after the infinite number of default waves,

$$q_\infty = N(\delta_\infty), \quad \delta_\infty = \lim_{k \rightarrow \infty} \delta_k \quad (18)$$

and δ_∞ satisfies the fixed-point equation

$$x = F(x) \quad (19)$$

which can be rewritten as

$$f(x) = \delta_1, \quad f(x) = x - \kappa N(x) \quad (20)$$

Function $f(x)$ is monotonically increasing for $\kappa < \kappa_0$, where

$$\kappa_0 = \sqrt{2\pi} \simeq 2.5066 \quad (21)$$

so in this case there exists only one fixed point for any value of δ_1 . In the opposite case $\kappa > \kappa_0$, however, $f(x)$ has a minimum at x_0 and a maximum at $x_1 = -x_0$, where

$$x_0 = \sqrt{2 \ln \frac{\kappa}{\kappa_0}} \quad (22)$$

Consequently, in the interval

$$y_0 < \delta_1 < y_1, \quad y_0 = f(x_0), \quad y_1 = f(x_1) \quad (23)$$

there are three fixed points, z_1 , z_2 and z_3 , such that

$$z_1 < x_1 < z_2 < x_0 < z_3 \quad (24)$$

Fixed points z_1 and z_3 are stable, while z_2 is unstable, since

$$F'(z_1) < 1, \quad F'(z_2) > 1, \quad F'(z_3) < 1 \quad (25)$$

(the prime symbol denotes the first derivative of a function), so the choice is between z_1 and z_3 . The basins of attraction are $(-\infty, z_2)$ for z_1 and $(z_2, +\infty)$ for z_3 ; the initial value of the orbit is δ_1 , and $\delta_1 < z_1 < z_2$, which means that δ_1 belongs to the basin of attraction of the leftmost fixed point z_1 . As a result, $\delta_\infty = z_1$ for the above interval of δ_1 values. The overall picture is as follows: for any $\delta_1 < y_0$, there is a single fixed point; when δ_1 reaches y_0 , a stable-unstable pair of fixed points, z_2 and z_3 , is born at $x = x_0$ (this event is called the fold bifurcation, e.g., [8]), but $\delta_\infty = z_1$ until the value $\delta_1 = y_1$ is reached. At this level of δ_1 , fixed points z_1 and z_2 collide and annihilate each other at $x = x_1$ (another fold bifurcation). Function $g(x)$, linking the total loss q_∞ and the direct loss q_1 via the relation between δ_∞ and δ_1 ,

$$\delta_\infty = g(\delta_1) \quad (26)$$

jumps therefore from $z_1 = z_2 = x_1$ to the value $z_3 = x_2 \neq x_1$ of the rightmost fixed point which is found from the condition $f(x_2) = y_1$. Functions $g_k(x)$, defined as

$$\delta_k = g_k(\delta_1) \quad (27)$$

are, on the other hand, continuous, increasing and can be inverted, leading to

$$\delta_1 = h_k(\delta_k), \quad h_k(x) = g_k^{-1}(x) \quad (28)$$

Functions $h_k(x)$ are also continuous and increasing; together with their limiting function

$$h(x) = \lim_{k \rightarrow \infty} h_k(x) \quad (29)$$

they are used below to calculate loss distributions for q_k and q_∞ . Function $h(x)$ coincides with $f(x)$ for $\kappa < \kappa_0$ and is constructed in the case $\kappa > \kappa_0$ by replacing the segment of $f(x)$ between x_1 and x_2 by the horizontal line connecting points (x_1, y_1) and (x_2, y_1) (this is the only possibility for the limit of a sequence of increasing functions which includes both these points).

The loss after k default waves q_k is, via Eqs. (8), (16) and (27), a function of the market factor Z which is a standard normal variable. The probability distribution for q_k is found by deriving an inequality for Z equivalent to the inequality $q_k < x$. Since $N(x)$ in Eq. (16) is an increasing function, the inequality for δ_k reads

$$\delta_k < N^{-1}(x) \quad (30)$$

Function $h_k(x)$ in Eq. (28) is also increasing, so the inequality for δ_1 is

$$\delta_1 < h_k(N^{-1}(x)) \quad (31)$$

Finally, because of Eq. (8), the equivalent inequality for Z is

$$Z > -A_k(x), \quad A_k(x) = \frac{1}{\sqrt{\rho}} \left[\sqrt{1-\rho} h_k(N^{-1}(x)) - N^{-1}(q) \right] \quad (32)$$

As a result, the CDF for q_k is

$$F_k(x) = P(q_k < x) = P(Z > -A_k(x)) = N(A_k(x)) \quad (33)$$

and the corresponding probability density function (PDF) has the form

$$p_k(x) = F'_k(x) = \sqrt{\frac{1-\rho}{\rho}} h'_k(N^{-1}(x)) \frac{\phi(A_k(x))}{\phi(N^{-1}(x))} \quad (34)$$

where $\phi(x) = N'(x)$ is the standard normal PDF.

The probability distribution for the total loss q_∞ is obtained as the limit of this result when $k \rightarrow \infty$ and corresponds to using $h(x)$ instead of $h_k(x)$ in Eqs. (32)-(34). For the purpose of completeness,

$$F_\infty(x) = P(q_\infty < x) = N(A_\infty(x)) \quad (35)$$

$$p_\infty(x) = F'_\infty(x) = \sqrt{\frac{1-\rho}{\rho}} h'(N^{-1}(x)) \frac{\phi(A_\infty(x))}{\phi(N^{-1}(x))} \quad (36)$$

$$A_\infty(x) = \frac{1}{\sqrt{\rho}} \left[\sqrt{1-\rho} h(N^{-1}(x)) - N^{-1}(q) \right] \quad (37)$$

From the properties of $h(x)$ it follows that in the case $\kappa > \kappa_0$ its first derivative $h'(x)$ vanishes for $x_1 < x < x_2$ and is discontinuous at $x = x_2$ (it is continuous at $x = x_1$, since $f'(x_1) = 0$). This translates to the PDF $p_\infty(x)$ which in this regime is split into two parts (it is equal to zero when $N(x_1) < x < N(x_2)$) and has a jump at $x = N(x_2)$. The first of these features is consistent with the

inability of δ_∞ to have any value in the interval (x_1, x_2) , which is reflected in the jump of $g(x)$.

The probability distribution for the direct loss q_1 , on the other hand, corresponds to $h_1(x) = x$ and has the form

$$F_1(x) = P(q_1 < x) = N(A_1(x)) \quad (38)$$

$$p_1(x) = F'_1(x) = \sqrt{\frac{1-\rho}{\rho}} \frac{\phi(A_1(x))}{\phi(N^{-1}(x))} \quad (39)$$

$$A_1(x) = \frac{1}{\sqrt{\rho}} \left[\sqrt{1-\rho} N^{-1}(x) - N^{-1}(q) \right] \quad (40)$$

This is the well-known Vasicek distribution [9]. Because of the similarity in the mathematical structure of the two distributions and the fact that Eqs. (38)-(40) are obtained from Eqs. (35)-(37) in the limit $\kappa \rightarrow 0$, the derived total loss distribution can be considered as a modification of the Vasicek distribution which takes into account the systemic component of the network loss in the proposed model.

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